b. Is the function $f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \in]0,1[\\ 0, & x = 0 \end{cases}$ Riemann integrable? Evaluate the integral of $f(x)$ on $[0,1]$.

c. Is a monotonic function f on $[a,b]$ Riemann integrable? Show that $f(x) = \begin{cases} \frac{1}{2^n}, & when\ \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, (n = 0,1,2,\ldots\ldots) \\ 0, & x = 0 \end{cases}$ is monotonous on $[0,1]$. Find the value of $\int_0^1 f(x)dx$.

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M.Sc. MATHEMATICS
FIRST SEMESTER
ANALYSIS-I
MSM-101

(Use separate answer scripts for Objective & Descriptive)

Duration: 3 hrs. **Full Marks: 70**

(PART-A : Objective)

Time: 20 min. **Marks: 20**

Choose the correct answer from the following: ***1×20=20***

1. The metric space (X,d) with $d(x,y) = 1$ for $x \neq y$ and $d(x,y) = 0$ for $x = y$ is:
   a. Unique
   b. Regular
   c. Discrete
   d. Euclidean

2. In the metric space (R,d) with the usual metric d the radius of the open sphere $S_r(a)$ is:
   a. a
   b. r
   c. d
   d. Infinite

3. In the set [0, 1] the point 1 is:
   a. An isolated point
   b. A limit point
   c. A point in the set
   d. None of these

4. The derived set of every subset A of a discrete metric space:
   a. Contains an infinite number of points
   b. Is empty
   c. Is discrete
   d. Is A itself

5. The Cantor set at the nth step is the union of n number of:
   a. Closed sets
   b. Open sets
   c. Singleton sets
   d. Intervals

6. If the closure of any subset A of a metric space (X,d) is given by $\bar{A} = X$, then it is:
   a. Null
   b. Dense
   c. Singleton
   d. None of the above

7. If A is a dense-in-itself set a metric space (X,d) then:
   a. $A \subseteq A'$
   b. $A = \bar{A}$
   c. int $(\bar{A}) = \Phi$
   d. A is perfect

8. A set X is a compact metric space with the metric d if:
   a. X is finite and d is discrete.
   b. X is infinite and d is discrete.
   c. X is the set of reals and d is the usual metric.
   d. $X =]0,1]$ and d is the usual metric.

9. The metric space $(X,d)$with the usual metric d and $X =]0,1]$ becomes complete if:
   a. d becomes discrete
   b. rational part of X is taken out
   c. '0' is adjoined to X
   d. $d(x,y) = \int_0^1 |x(t) - y(t)|\, dt$

10. The domain of a sequence in a metric space (S,d) with range S is:
   a. $\mathbb{R}$
   b. Q
   c. N
   d. S

11. Every bounded sequence in $\mathbb{R}^n$:
a. Has a convergent subsequence
b. Covers an infinite set
c. Is oscillatory
d. Is divergent

12. The subsequence $\{1, 1, 1, 1, \ldots\ldots\ldots\}$, of the sequence $\{1, -1, 1, -1, \ldots\ldots\ldots\}$ is:
a. Finite
b. Divergent
c. Oscillatory
d. Convergent

13. A Cauchy sequence in a metric space (X, d) is defined for:
a. An arbitrary number
b. A discrete metric
c. A preassigned small positive number
d. A positive real number

14. In a Cauchy sequence $\{a_n\}$ of points of the metric space (X, d) the metric d satisfies the condition $d(m, n) < \varepsilon\ \forall\ m, n \geq n_0$ for each $\varepsilon > 0$, where n_0 is:
a. An element of X
b. An index of metric
c. A positive integer
d. A real number

15. The radius of convergence of the power series $1 + x^2 + x^4 + x^6 + \ldots\ldots\ldots$ is equal to:
a. 0
b. 1
c. $\frac{2}{3}$
d. ∞

16. If $f(x) = x$ for rational values of x and $f(x) = 0$ for irrational values of x, then f is:
a. A piecewise continuous function
b. Not defined at at $x = 0$
c. Continuous at $x = 0$
d. Discontinuous at $x = 0$

17. If (X, d_1) and (Y, d_2) are any two metric spaces, then the constant function $f: X \to Y$ is:
a. Continuous on X
b. Not continuous on X
c. Has the value 0
d. Has the value $Max.(d_1, d_2)$

18. In case of isometry between two metric spaces, the isometry is:
a. Into
b. Onto
c. One-to-one
d. Many-one

19. If $f_1, f_2, \ldots\ldots\ldots, f_k$ are the components of the vector valued function f then f' at a point x is equal to:
a. $\sum_1^k \frac{\partial}{\partial x} f_k$
b. $\frac{\partial}{\partial x} \sum_1^k f_k$
c. $(f'_1, f'_2, f'_3, \ldots\ldots\ldots, f'_k)$
d. $\frac{\partial}{\partial x} f(f_1, f_2, \ldots\ldots\ldots, f_k)$

20. In the Riemann integral of the bounded function f over $[a, b]$:
a. The upper integral is defined.
b. The lower integral is defined.
c. Both the upper and lower integrals are defined.
d. The upper and lower integrals are unequal.

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(PART-B : Descriptive)

Time : 2 hrs. 40 min. Marks : 50

[Answer question no.1 & any four (4) from the rest]

1. a. If (X, d) be a metric space, then show that the mapping $\rho: X \times X \to R$ defined by $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}\ \forall\ x, y \in X$ is a metric on X. 5+5=10
b. Show that d and ρ are equivalent metrics in the above metric space(X, d).

2. a. Define limit point, isolated point and derived set of a subset A of the metric space (X, d). Does the set of integers I possess a limit point? 4+6=10
b. Explain the terms open sphere, closed sphere and neighbourhood of a point $a \in X$ of the metric space (X, d).

3. a. Construct the Cantor set by considering ternary operation to the 8th stage. 4+6=10
b. Show that the Cantor set is a perfect set.

4. a. Prove that the closure $\bar{A}$ of any subset A of a metric space (X, d) is a closed set. 4+3+3=10
b. Show that: (i) $\left\{\frac{(-1)^{n-1}}{n!}\right\}, n \in N$, converges to zero (ii) $\{n(-1)^n\}, n \in N$, oscillates infinitely.
c. Define subsequence of a sequence $\{S_n\}, n \in N$. Give example to show that a bounded sequence contains a convergent subsequence.

5. a. Prove that every closed subset of a compact metric space is compact. 4+6=10
b. Prove that a metric space (X, d) is sequentially compact if and only if it has the Bolzano-Weierstrass property.

6. a. Show that the space $C[0,1]$ of all continuous bounded real-valued functions defined on the closed interval $[0,1]$ with the metric $d(f, g) = \max\{|f(x) - g(x)| : 0 \leq x \leq 1\}$ is complete. 4+6=10
b. State and prove D' Alemberts Ratio test for convergence of series of positive terms.

7. a. Define homeomorphism and isometry between two metric spaces. Prove that isometric metric spaces are homeomorphic. 5+5=10
b. Prove that any contraction mapping f of a non-empty complete metric space (X, d) into itself has a unique fixed point.

8. a. State the Fundamental Theorem of Calculus for a bounded and integrable function f on $[a, b]$. 3+3+4=10

PTO