M.Sc. MATHEMATICS
THIRD SEMESTER
FUNCTIONAL ANALYSIS
MSM-302

(Use separate answer scripts for Objective & Descriptive)

Duration : 3 hrs. **Full Marks : 70**

(PART-A : Objective)

Time : 20 min. Marks : 20

Choose the correct answer from the following: ***1×20=20***

1. Which of the following is false?
   - a. Every vector space is a metric space.
   - b. Every normed linear space is a metric space.
   - c. Every metric space is a normed linear space.
   - d. R^n is a banach space w.r.t. any norm.

2. A non-empty subset of a normed space is compact if it is:
   - a. Closed
   - b. Complete
   - c. Bounded
   - d. Both (a) and (c)

3. A linear map from a normed space X to Y is continuous iff T is:
   - a. Bounded
   - b. Convergent
   - c. Continuous
   - d. Complete

4. Which of the following space is not complete?
   - a. *The space* l^p
   - b. *the space* l^∞
   - c. The space C[0,1] of continuously differentiable function over the norm $\|x\|_1 = \int_{-1}^{1} |x(t)|$ is complete.
   - d. None.

5. Which of the following is false?
   - a. The set $B(X,Y)$ of all bounded linear operators from X *into* Y is a subspace of the space $L(X,Y)$.
   - b. A finite dimensional normed space is not a banach space.
   - c. All norms on finite dimensional linear space are equivalent.
   - d. None.

6. Which of the following is/are examples of commutative banach algebra?
   - a. R
   - b. C
   - c. Both R *and* C
   - d. None

7. Two norms $\|.\|_1$ *and* $\|.\|_2$ are said to be equivalent if $\exists\, m, M > 0$ *such that*:
   - a. $m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$
   - b. $m\|x\|_1 \geq \|x\|_2 \geq M\|x\|_2$
   - c. $m\|x\|_1 \geq \|x\|_2 \leq M\|x\|_2$
   - d. $m\|x\|_1 \leq \|x\|_2 \geq M\|x\|_2$

8. Which of the following statement is true?
   - a. A real inner product space is conjugate symmetric.
   - b. A complex inner product space is linear in the second argument.
   - c. A complex inner product space is symmetric.
   - d. A real inner product space is linear in the first argument.

9. Borel-Lebesgue theorem deals with which property?
   a. Boundedness b. Completeness
   c. Compactness d. None

10. A linear map T is said to be invertible if:
   a. T is continuous b. T^{-1} exists
   c. Both (a) and (b) d. None

11. Which of the following is true?
   a. An inner product space is not a normed linear space.
   b. Any finite dimensional normed linear space is a banach space.
   c. The function space C[a,b] is not complete.
   d. None.

12. The Riesz representation theorem is not true in an inner product space which is:
   a. Complete b. Not complete c. Compact d. None

13. Which of the following statement is false?
   a. In a Hilbert space the norm induced by the inner product satisfies the parallelogram law.
   b. In l_n^1 space, where $n > 1$ the parallelogram law not is true.
   c. In l_n^2 space, where $n > 1$ the parallelogram law is true.
   d. A complete inner product space is called Hilbert space.

14. Which of the following is false?
   a. Differential operators are unbounded b. Integral operators are bounded
   c. Differential operators are bounded d. None

15. Which of the following statement is true?
   a. Any two norms in a finite dimensional space is always equivalent.
   b. An inner product space is not a normed space.
   c. Any two norms in an infinite dimensional space is always equivalent.
   d. None

16. The principle of uniform boundedness is also known as:
   a. Closed graph theorem b. Riesz theorem
   c. Open mapping theorem d. Banach-steinhaus theorem

17. Which of the following is known as Bessel's inequality for finite case?
   a. $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \le \|x\|^2$ b. $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 > \|x\|^2$
   c. $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \ge \|x\|^2$ d. $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 = \|x\|^2$

18. A norm in an inner product space is defined by:
   a. $\|x\|^2 = \sqrt{\langle x, x\rangle}$ b. $\|x\| = \sqrt[3]{\langle x, y\rangle}$
   c. $\|x\|^3 = \sqrt{\langle x, x\rangle}$ d. $\|x\| = \sqrt{\langle x, x\rangle}$

19. Two vectors x *and* y are said to be orthogonal if:
   a. $\langle x . y\rangle = 1$ b. $\langle x . y\rangle = 2$
   c. $\langle x . y\rangle - 0$ d. None

20. Find the correct statement.
   a. The Hahn-Banach theorem deals with extension of linear functional.
   b. The series $\sum_{n=1}^{\infty} |\langle x, e_i\rangle|^2$ is divergent.
   c. Orthonormal sets are incomplete.
   d. None.

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(PART-B :Descriptive)

Time: 2 hrs. 40min. Marks: 50

[Answer question no.1 & any four (4) from the rest]

1. State and prove Riesz-representation theorem. Also prove that this theorem is not true in an incomplete inner product space. 7+3=10

2. a. Given that T is a linear operator such that T: X $\xrightarrow{onto}$ Y, both X and Y are normed spaces. Prove that T^{-1} exists and and is a bounded linear operator iff $\exists$ *a constant* $K > 0$ such that $\|Tx\|_Y \ge K \|x\|_Y, \forall x \in X$. 5+5=10
   b. Prove that *the linear space* $C[a, b]$ of all continuous functions defined
   on [a, b] is a banach space.

3. a. Given that Y is any subspace in a normed linear space $(X, \|.\|)$. Show that $\bar{Y}$ is a closed subspace. 4+6=10
   b. State Hahn-Banach theorem. If N is a normed linear space and x_0 is a non-zero vector in N, then prove that there exists a functional f_0 in N^* *such that* $f_0(x_0) = \|x_0\|$ *and* $\|f_0\| = 1$.

4. a. Given M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M. If d is the distance from x_0 to M, then prove that there exists a functional $f_0 \in N^*$ such that $f_0(M) = 0,\ f_0(x_0) = d$ *and* $\|f_0\| = 1$. 6+4=10
   b. Write the statements of open mapping theorem and closed graph theorem.

5. a. Define Isomorphism in inner product space. Prove that in an inner product space if $x_n \to x$ *and* $y_n \to y$, *then* $\langle x_n, y_n\rangle \to \langle x, y\rangle$, as $n \to \infty$. 5+5=10
   b. If x *and* y are any two vectors in a Hilbert space H, then prove that $4\langle x, y\rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$.

6. a. State and prove Banach-Steinhaus Theorem. 6+4=10
   b. Prove that an inner product space is a normed linear space.

7. a. If H is a Hilbert space, then prove that H^* is also a Hilbert space with the inner product defined by $\langle f_x, f_y\rangle = \langle y, x\rangle$. 5+5=10
   b. Let T be an operator on a Hilbert space H. Then prove that there exists a unique operator T^* on H such that for all $x, y \in H, \langle Tx, y\rangle = \langle x, T^* y\rangle$.

8. a. Define orthonormal set in a Hilbert space. Let x *and* y be two orthogonal vectors in a Hilbert space H, then prove that $\|x + y\|^2 = \|x - y\|^2 = \|x\|^2 + \|y\|^2$. 1+2+3+4=10
   Show that the space l^p *with* $p \neq 2$ is not an inner product space.
   b. Derive Cauchy- Schwarz inequality from Bessel's inequality.

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