1 DEFINITION AND FUNDAMENTAL PROPERTIES

1.1 Two view-points of Integration.

There are two distinct view-points from which the process of integration can be considered. We may consider *integration as the inverse of differentiation* and make this as our starting point; or else, we may start with defining *integration as a certain summation* and then proceed to show that the results is identical with the reversal of a differentiation under certain conditions.

The establishment of an identity of the two view-points is referred to as the *Fundamental Theorem* of Integral Calculus. We shall consider both the points of view, starting first of all with the former, reserving the consideration of the latter for a subsequent chapter.

1.2 Integration, the inverse process of differentiation.

If $f(x)$ be a given function of x and if another function $F(x)$ be obtained such that its differential coefficient with respect to x is equal to $f(x)$, then $F(x)$ is defined as an *integral*, or more properly *an indefinite integral of* $f(x)$ *with respect to x.*

The process of finding an integral of a function of x is called *Integration* and the operation is indicated by writing the *integral sign** $\int$ before the given function and dx after the given function, the symbol dx indicating that x is *the variable of integration.* The function to be integrated, viz., $f(x)$ is called the *Integrand.*

Symbolically, if $\dfrac{d}{dx}F(x) = f(x)$,

then $\int f(x)dx = F(x)$,

where $\int f(x)dx$ is called an indefinite integral of $f(x)$ with respect to x.

It will be shown later that if f is condition then $\int f(x)\,dx$ exists.

Thus, considered as symbols of operation,

$$\frac{d}{dx}() \text{ and } \int () \, dx \text{ are inverse to each other.}$$

* Historically this sign is elongated S, the initial letter of the word 'sum', since integration was originally studied as a process of summation. (See Chapter VI.)

1.3. Constant of integration.

It may be noted that if $\frac{d}{dx}F(x) = f(x)$, then we also have

$$\frac{d}{dx}\{F(x)+C\} = f(x),$$

where C is an arbitrary constant.

Thus, if $\int f(x)dx = F(x)$, a general value of the indefinite integral

$$\int f(x)dx = F(x) + C.$$

In other words, in finding the indefinite integral of a function $f(x)$, *an arbitrary constant* is to be added to the result to make it general. This is the reason why the integral is referred to as an indefinite integral. The arbitrary constant is usually referred to as the *constant of integration.*

Addition of constant C to $F(x)$ and not any non-constant function is explained below.

We know that if two functions $\phi(x), \psi(x)$ defined on the closed interval $[a, b]$ are such that

$$\frac{d}{dx}[\phi(x)] = \frac{d}{dx}[\psi(x)],$$

for all $x \in [a, b]$ then $\phi(x) - \psi(x)$ is constant on $[a, b]$.

Thus, it is possible to get *the indefinite integral of the same function in different forms by different processes, but ultimately these forms can at most differ from each other by constant quantities only.*

Hence, an arbitrary constant added to the indefinite integral of a given function obtained by any process makes the result perfectly general.

In the following pages, we shall first of all deal with indefinite integrals. For the sake of convenience *the arbitrary constant of inregration has generally been omitted* but it is always understood to be present in every case, *and should be supplied by the students in the result.*

It should also be noted that in case an indefinite integral consists of a sum or difference of two or more integrals, the addition of one arbitrary constant for each integral is equivalent to the addition of a single arbitrary constant, denoting their algebraic sum, in the final result. For illustrations, see Illustrative Examples in Art. 1.7.

1.4. General laws satisfied by integrals.

(i) *The Integral of the sum or difference of any finite number of functions is equal to the sum or difference of the integrals of the functions taken separately.*

This follows immediately from the known results of the Differential Calculus. For we know that

$$\frac{d}{dx}\{f_1(x)+f_2(x)-f_3(x)\ldots\}=f_1'(x)+f_2'(x)-f_3'(x)+\ldots$$

$$\therefore \int\{f_1'(x)+f_2'(x)-f_3'(x)+\ldots\}dx$$

$$=f_1(x)+f_2(x)-f_3(x)+\ldots$$

$$=\int f_1'(x)\,dx+\int f_2'(x)\,dx-\int f_3'(x)\,dx+\ldots$$

[Since $\frac{d}{dx}f(x)=f'(x)$, $\therefore f(x)=\int f'(x)\,dx$, etc.].

(ii) *The operation of integration is commutative with regard to a constant, i.e.,* a factor of the integrand which is constant with regard to the variable of integration can be taken outside the sign of integration.

Symbolically, $\int Af(x)\,dx=A\int f(x)\,dx$.

This follows immediately from the fact that

$$\frac{d}{dx}\{AF(x)\}=A\frac{d}{dx}\{F(x)\}=Af(x)\text{, say,}$$

so that $\int Af(x)\,dx=AF(x)=A\int f(x)\,dx$,

since, by our supposition, $\frac{d}{dx}F(x)=f(x)$.

(iii) Combining the above two results, we can write

$$\int\{Af_1(x)\pm Bf_2(x)\pm Cf_3(x)+\ldots\}dx$$

$$=A\int f_1(x)\,dx\pm B\int f_2(x)\,dx\pm C\int f_3(x)\,dx+\ldots$$

1.5. Fundamental Integrals.

A slight acquaintance with the Differential Calculus will at once suggest the integrals in many elementary cases. As the first step towards the facility in integration, the student must be thoroughly familiar with the *fundamental integrals* (shown in the next page) :

(i) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$.

Cor. $\int \dfrac{dx}{x^n} = -\dfrac{1}{(n-1)x^{n-1}}$ $(n \neq 1)$.

$\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$

$\int dx = x$.

(ii) $\int \dfrac{dx}{x} = \log|x|$ if $x \neq 0$.

(iii) $\int e^{mx}\,dx = \dfrac{e^{mx}}{m}$ if $m \neq 0$.

Cor. $\int e^x\,dx = e^x$.

$\int a^x\,dx = \dfrac{a^x}{\log_e a}$ $(a > 0, a \neq 1)$.

(iv) $\int \sin mx\,dx = -\dfrac{\cos mx}{m}$ if $m \neq 0$.

Cor. $\int \sin x\,dx = -\cos x$.

(v) $\int \cos mx\,dx = \dfrac{\sin mx}{m}$.

Cor. $\int \cos x\,dx = \sin x$.

(vi) $\int \sec^2 mx\,dx = \dfrac{\tan mx}{m}$ if $m \neq 0$.

Cor. $\int \sec^2 x\,dx = \tan x$.

(vii) $\int \operatorname{cosec}^2 mx\,dx = -\dfrac{\cot mx}{m}$ if $m \neq 0$.

Cor. $\int \operatorname{cosec}^2 x\,dx = -\cot x$.

(viii) $\int \sec mx \tan mx\,dx = \dfrac{\sec mx}{m}$ if $m \neq 0$.

Cor. $\int \sec x \tan x\,dx = \sec x$.

(ix) $\int \operatorname{cosec} mx \cot mx dx = -\frac{\operatorname{cosec} mx}{m}$ if $m \neq 0$.

Cor. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$

(x) $\int \sinh mx\, dx = \frac{\cosh mx}{m}$ if $m \neq 0$.

Cor. $\int \sinh x\, dx = \cosh x$

(xi) $\int \cosh mx\, dx = \frac{\sinh mx}{m}$ if $m \neq 0$.

Cor. $\int \cosh x\, dx = \sinh x$

The results can be easily verified by differentiating the right side in each case.

Other standard results of integration which are also very useful for application will be found in the subsequent chapters.

Note 1. Since the integral (i) is frequently used, for the sake of covenience, we give here a concise verbal statement of the result, *viz.*, "*Increase the index by one and divide it by the increased index*".

Note 2. Since $\log x$ is real when $x > 0$ and $\frac{d}{dx}(\log x) = \frac{1}{x}$,

so, $\int \frac{1}{x} dx = \log x$ is difined for $x > 0$.

When $x < 0$, *i.e.*, $-x > 0$, $\frac{d}{dx} \log(-x) = \frac{-1}{-x} = \frac{1}{x}$.

Therefore when $x < 0$, $\int \frac{1}{x} dx = \log(-x)$. Hence both these results will be included if we write $\int \frac{1}{x} dx = \log|x|$.

In the formula and examples where integrals of this type occurs, *i.e.*, where the value of an integral involves the logarithm of a function and the function may become negative for some values of the varaible of the function, the absolute value sign $|\ |$ enclosing the function should be given, but it has generally been omitted, though it is always understood to be present and it should be supplied by the students.

Note 3. Different algebraical symbols *a*, *b*, *m*, *n*, etc. occurring in integrands are generally supposed to be different unless otherwise stated.

Note 4. In the above integrals (iii), (iv), (v), (x), (xi) it is tacitly assumed that m is a non-zero constant.

1.6. Standard methods of integration.

The different modes of integration all aim at reducing a given integral to one of the Fundamental or known integrals. As a matter of fact, there are two principal processes :

(i) *The method of substitution, i.e.*, a change of the independent variable.

(ii) *Integration by parts*.

In some cases, when the integrand is a rational fraction it may be broken into *partial fractions* by the rules of Algebra, and then each part may be integrated by one of the above methods. (See *Chapter V*)

In some cases of irrational functions, the method of Integration by rationalization is adopted, which is a special case of (i) above.

In some cases, integration by the method of *Successive Reduction* is resorted to, which really falls under case (ii). (*See Chapter IX*).

It may be noted that the classes of integrals which are reducible to one or other of the fundamental forms by the above processes are very limited, and that the large majority of the expressions, under proper restrictions, can only be integrated by the aid of *infinite series*, and in some cases when the integrand involves expressions under a radical sign containing powers of x beyond the second, the investigation of such integrals has necessitated the introduction of higher classes of transcendental function such as elliptic functions, etc.

In fact, integration is, on the whole, a more difficult operation than differentiation. The Differential Calculus gives general rules for differentiation, but Integral Calculus gives no such corresponding general rules for performing the inverse operation. Integration is essentially a tentative process. In fact, so simple an integral in appearance as

$$\int \sqrt{x}\cos x\,dx, \text{ or } \int \frac{\sin x}{x}\,dx$$

can not be worked out; that is there is no *elementary function* whose derivative is $\sqrt{x}\cos x$, or $(\sin x)/x$, though the integrals exist. There is quite a large number of integrals of these types.

1.7. Illustrative Examples.

Ex. 1. Integrate $\int \sin^2 x\,dx$.

$$I = \int \frac{1}{2}(1 - \cos 2x)\,dx$$

$$= \int \frac{1}{2}\,dx - \int \frac{1}{2}\cos 2x\,dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Ex. 2. Integrate $\int \tan^2 x\,dx$.

$$I = (\sec^2 x - 1)\,dx$$

$$= \int \sec^2 x\,dx - \int dx = \tan x - x + C$$

Ex. 3. Integrate $\int \frac{5(x-3)^2}{x\sqrt{x}}\,dx$.

$$I = \int \frac{5x^2 - 30x + 45}{x\sqrt{x}}\,dx$$

$$= 5\int x^{\frac{1}{2}}\,dx - 30\int \frac{dx}{\sqrt{x}} + 45\int x^{-\frac{3}{2}}\,dx$$

$$= 5.\frac{2}{3}x^{\frac{3}{2}} - 30.2\sqrt{x} + 45(-2x^{-\frac{1}{2}}) + C$$

$$= \frac{10}{3}x^{\frac{3}{2}} - 60\sqrt{x} - 90x^{-\frac{1}{2}} + C.$$

Ex. 4. Integrate $\int \sin 3x \cos 2x\,dx$.

$$\sin 3x \cos 2x = \frac{1}{2}.2\sin 3x \cos 2x = \frac{1}{2}(\sin 5x + \sin x)$$

$$\therefore\ I = \frac{1}{2}[\int \sin 5x\,dx + \int \sin x\,dx]$$

$$= \frac{1}{2}[-\frac{1}{5}\cos 5x - \cos x] + C$$

Note. Henceforth the arbitrary constant of integration will be omitted in the illustrative examples, as also in the answers to the set of examples.

1.8 Miscellaneous Workedout Examples

Integrate the following :

Ex. 1. $\int (\tan x + \cot x)^2\, dx$

Solution : $\int (\tan x + \cot x)^2\, dx$

$$= \int (\tan^2 x + \cot^2 x + 2)\, dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$$

$$= \tan x - \cot x + C$$

$$= \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} + C$$

$$= \frac{-(\cos^2 x - \sin^2 x)}{\sin x \cos x} + C = -2\cot 2x + C.$$

Ex. 2. $\int \cos(\pi\theta + 1)\, d\theta$

Solution : $\int \cos(\pi\theta + 1)\, d\theta$

$$= \int (\cos \pi\theta \cos 1 - \sin \pi\theta \sin 1)\, d\theta$$

$$= \cos 1 \int \cos \pi\theta\, d\theta - \sin 1 \int \sin \pi\theta\, d\theta$$

$$= \cos 1 \cdot \frac{\sin \pi\theta}{\pi} + \sin 1 \frac{\cos \pi\theta}{\pi} + C$$

$$= \frac{1}{\pi} \sin(\pi\theta + 1) + C.$$

Ex. 3. $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\, dx$ [C. P. 1994, '99]

Solution : $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\, dx$

$$= \int \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x - 2\sin 3x \cos 3x}\, dx$$

$$= \int \frac{2\sin\frac{3x}{2}\cdot\cos\frac{3x}{2}\cdot 2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}}{\sin 3x - \sin 6x}\,dx$$

$$= \int \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}}\,dx$$

$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}\,dx$$

$$= -\int(\cos 2x + \cos x)\,dx$$

$$= -\left(\frac{1}{2}\sin 2x + \sin x\right) + C \quad = -\sin x(1+\cos x) + C.$$

Ex. 4. $\int \cos x\cdot\cos 2x\cos 3x\,dx$

Solution : $\int \cos x\cdot\cos 2x\cos 3x\,dx$

$$= \frac{1}{2}\int \cos x(2\cos 3x\cos 2x)\,dx$$

$$= \frac{1}{2}\int \cos x(\cos 5x + \cos x)\,dx$$

$$= \frac{1}{4}\int(2\cos 5x\cos x + 2\cos^2 x)\,dx$$

$$= \frac{1}{4}\int(\cos 6x + \cos 4x + \cos 2x - 1)\,dx$$

$$= \frac{1}{4}\left(\frac{1}{6}\sin 6x + \frac{1}{4}\sin 4x + \frac{1}{2}\sin 2x - x\right) + C$$

$$= \frac{1}{48}(2\sin 6x + 3\sin 4x + 6\sin 2x - 12x) + C.$$

Ex. 5. $\int \phi(x)\,dx$, where $\phi(x) = \tan^{-1}\left\{\dfrac{\log\left(\frac{e}{x^4}\right)}{\log(ex^2)}\right\} - \tan^{-1}\left\{\dfrac{\log(ex^6)}{\log\left(\frac{e^3}{x^2}\right)}\right\}$