

Solution : We have,

$$\begin{aligned}\tan^{-1} \left\{ \frac{\log \left(\frac{e}{x^4} \right)}{\log (ex)^2} \right\} &= \tan^{-1} \left\{ \frac{\log e - \log x^4}{2 \log (ex)} \right\} \\ &= \tan^{-1} \left\{ \frac{1 - 4 \log x}{2(\log e + \log x)} \right\} \\ &= \tan^{-1} \left\{ \frac{\frac{1}{2} - 2 \log x}{1 + \frac{1}{2} \cdot 2 \log x} \right\} = \tan^{-1} \frac{1}{2} - \tan^{-1}(2 \log x)\end{aligned}$$

and

$$\begin{aligned}\tan^{-1} \left\{ \frac{\log (ex^6)}{\log \left(\frac{e^3}{x^2} \right)} \right\} &= \tan^{-1} \left\{ \frac{\log e + \log x^6}{\log e^3 - \log x^2} \right\} \\ &= \tan^{-1} \left\{ \frac{1 + 6 \log x}{3 - 2 \log x} \right\} \\ &= \tan^{-1} \left\{ \frac{\frac{1}{3} + 2 \log x}{1 - \frac{1}{3} \cdot 2 \log x} \right\} = \tan^{-1} \frac{1}{3} + \tan^{-1}(2 \log x)\end{aligned}$$

$$\begin{aligned}\therefore \int \phi(x) dx &= \int \left\{ \tan^{-1} \frac{1}{2} - \tan^{-1}(2 \log x) + \left(\tan^{-1} \frac{1}{3} + \tan^{-1}(2 \log x) \right) \right\} dx \\ &= \int \left(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} \right) dx \\ &= \int \tan^{-1}(1) dx \\ &= \int \tan^{-1} \left(\tan \frac{\pi}{4} \right) dx \\ &= \frac{\pi}{4} \int 1 \cdot dx = \frac{\pi x}{4} + C.\end{aligned}$$

Ex. 6. Find $\phi(x)$, if $\phi(0) = \phi'(0) = 0$ and $\phi''(x) = \cos^2 x + 5$.

Solution : $\because \phi''(x) = \cos^2 x + 5$

$$\begin{aligned}\phi'(x) &= \int (\cos^2 x + 5) dx \\ &= \int \left\{ \frac{1}{2}(1 + \cos 2x) + 5 \right\} dx = \frac{1}{2} \cdot \frac{1}{2} \sin 2x + \frac{11x}{2} + A,\end{aligned}$$

where A is the constant of integration.

$$\because \phi'(0) = 0, \quad \frac{1}{4} \cdot \sin 0 + \frac{11}{2} \times 0 + A, \quad \text{i.e., } A = 0$$

$$\therefore \phi'(x) = \frac{1}{4} \cdot \sin 2x + \frac{11}{2} \times x + A$$

$$\begin{aligned}\phi(x) &= \int \phi'(x) dx \\ &= \int \left(\frac{1}{4} \sin 2x + \frac{11x}{2} \right) dx \\ &= -\frac{1}{8} \cos 2x + \frac{11}{4} x^2 + B,\end{aligned}$$

where B is the constant of integration.

$$\because \phi(0) = 0,$$

$$0 = -\frac{1}{8} \cos 0 + \frac{11}{4} \times 0 + B \quad \text{i.e., } B = \frac{1}{8}.$$

$$\text{Hence, } \phi(x) = \frac{1}{8}(1 + 22x^2 - \cos 2x).$$

Ex. 7. Show that $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx = -\frac{2}{\sin 2x} + C$.

$$\begin{aligned}\text{Solution : } I &= \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x} dx = \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} \\ &= \int \operatorname{cosec}^2 x dx - \int \sec^2 x dx = -\cot x - \tan x + C \\ &= -\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} + C = -\frac{\cos^2 x + \sin^2 x}{\sin x \cos x} + C \\ &= -\frac{2}{2 \sin x \cos x} + C = -\frac{2}{\sin 2x} + C = -2 \operatorname{cosec} x + C.\end{aligned}$$

Ex. 8. Find $\int \frac{\cos 2x - \cos 2\alpha}{\sin x - \sin \alpha} dx$.

$$\begin{aligned} \text{Solution : } I &= \int \frac{(1 - 2\sin^2 x) - (1 - 2\sin^2 \alpha)}{\sin x - \sin \alpha} dx \\ &= -2 \int \frac{\sin^2 x - \sin^2 \alpha}{\sin x - \sin \alpha} dx \\ &= -2 \int (\sin x + \sin \alpha) dx = 2 \cos x - 2x \sin \alpha + C. \end{aligned}$$

Ex. 9. Find

$$\int \left\{ \frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x \cos^2 x} + \sqrt{1 + \sin 2x} + \frac{1 + 2\sin x}{\cos^2 x} + \frac{1 - \cos 2x}{1 + \cos 2x} \right\} dx.$$

Solution :

$$\begin{aligned} I &= \int \left\{ \frac{5}{2} \operatorname{cosec} x \cot x + \sec x \tan x + \cos x + \sin x + \sec^2 x \right. \\ &\quad \left. + 2 \sec x \tan x + \tan^2 x \right\} dx \\ &= \int \left\{ \frac{5}{2} \operatorname{cosec} x \cot x + 3 \sec x \tan x + \cos x + \sin x + 2 \sec^2 x - 1 \right\} dx \\ &= -\frac{5}{2} \operatorname{cosec} x + 3 \sec x + \sin x - \cos x + 2 \tan x - x + C. \end{aligned}$$

Ex. 10. Find $\int \left\{ \frac{1-x^2}{1-x} - \frac{6}{x\sqrt{x^2-4}} + (\tan x + \cot x) \right\} dx$.

Solution :

$$\begin{aligned} I &= \int \left\{ \frac{(1-x)(1+x)(1+x^2)}{1-x} - \frac{6}{x\sqrt{x^2-4}} + \tan^2 x + \cot^2 x + 2 \right\} dx \\ &= \int \left\{ 1 + x + x^2 + x^3 - 6 \times \frac{1}{x\sqrt{x^2-1}} + \sec^2 x + \operatorname{cosec}^2 x \right\} dx \\ &= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{6}{2} \sec^{-1} \left(\frac{1}{2}x \right) + \tan x - \cot x + C \end{aligned}$$

EXAMPLES 1

Integrate the following:-

1. (i) $\int \frac{(1+x)^3}{x} dx$. (ii) $\int \sqrt{x} \left(x^5 + \frac{3}{x} \right) dx$.

2. (i) $\int \cos^2 x dx$. (ii) $\int \frac{\tan x}{\cot x} dx$.

(iii) $\int \frac{1 - \tan^2 x}{1 + \tan^2 x} dx$. (iv) $\int \frac{1 + \tan^2 x}{1 + \cot^2 x} dx$.

3. $\int \sec x (\sec x - \tan x) dx$.

4. (i) $\int \cos^2 ax dx$. (ii) $\int \cot^2 x dx$.

5. (i) $\int \frac{2e^{2x} + 3e^{4x} + 4}{e^{3x}} dx$. (ii) $\int \frac{e^{3x} + e^{5x}}{e^{-x} + e^{-x}} dx$.

(iii) $\int \frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}} dx$.

6. $\int (e^{a \log x} + e^{x \log a}) dx$.

7. (i) $\int \left(x\sqrt{x} - \frac{1}{3}\sqrt{x} + \frac{11}{\sqrt{x}} \right) dx$. (ii) $\int \frac{1-x^6}{1-x} dx$.

8. (i) $\int (a^{\frac{2}{3}} - x^{\frac{2}{3}})^3 dx$. (ii) $\int (x+2)(x+3)^2 dx$.

9. (i) $\int \frac{a^x + a^{2x} + a^{3x}}{a^{4x}} dx$. (ii) $\int \frac{8^{1+x} + 4^{1-x}}{2x} dx$.

10. (i) $\int \frac{(1-2x^2)^2}{x^3 \sqrt{x}} dx$. (ii) $\int \frac{a \sin^3 x + b \cos^3 x}{\sin^2 x \cos^2 x} dx$.

(iii) $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$.

11. (i) $\int \frac{\sin x + \operatorname{cosec} x}{\tan x} dx$. (ii) $\int \cos x^a dx$.

$$12. \int \frac{\operatorname{cosec} x + \tan^2 x + \sin^2 x}{\sin x} dx.$$

$$13. (i) \int \frac{x^3 - 4x^2 + 5x - 2}{x^2 - 2x + 1} dx.$$

$$(ii) \int \frac{x^3 - 6x + 9}{x + 3} dx.$$

$$14. (i) \int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx.$$

$$(ii) \int \frac{\cos x - \cos 2x}{1 - \cos x} dx.$$

$$(iii) \int \frac{\sec 2x - 1}{\sec 2x + 1} dx.$$

$$(iv) \int \frac{dx}{\cosh x + \sinh x}.$$

$$15. (i) \int \sqrt{1 + \sin x} dx.$$

$$(ii) \int \sqrt{1 - \sin x} dx.$$

$$[1 \pm \sin x = (\sin \frac{1}{2} x \pm \cos \frac{1}{2} x)^2]$$

$$16. \int \frac{\cos x - \sin x}{\cos x + \sin x} (2 + 2\sin 2x) dx.$$

$$17. (i) \int \sqrt{1 + \cos x} dx.$$

$$(ii) \int \sqrt{1 - \cos x} dx.$$

$$(iii) \int (3\sin x \cos^2 x - \sin^3 x) dx.$$

$$18. (i) \int \frac{dx}{1 + \sin x}.$$

$$(ii) \int \frac{dx}{1 + \cos x}.$$

[(i) Multiply numerator and denominator by $1 - \sin x$.]

$$19. (i) \int \frac{\cos 2x - \cos 2a}{\cos x - \cos a} dx.$$

$$(ii) \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx.$$

[(ii) Multiply numerator and denominator by $\sin 3x$.]

$$20. (i) \int \frac{dx}{\sin^2 x \cos^2 x} \quad [Put \sin^2 x + \cos^2 x \text{ in the numerator.}]$$

$$(ii) \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx.$$

$$(iii) \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx.$$

$$(iv) \int \frac{\cos^4 x - \sin^4 x}{\sqrt{1 + \cos 4x}} dx.$$

21. $\int \frac{\cos x}{\sin^2 x} (1 - 3\cos^3 x) dx$.

22. (i) $\int \cos^3 x dx$. (ii) $\int \sin^4 x dx$.

[(i) write $\cos^3 x = \frac{1}{4}(\cos 3x + 3 \cos x)$. (For another method see § 4.3.)]

23. (i) $\int \sin mx \sin nx dx$. (ii) $\int \cos 2x \cos 3x dx$.

24. (i) $\int \sin^2 x \cos^2 x dx$. (ii) $\int \sin^2 x \cos 2x dx$.

25. $\int \sin x \sin 2x \sin 3x dx$.

ANSWERS

1. (i) $\log x + 3x + \frac{3}{2}x^2 + \frac{1}{3}x^3$. (ii) $\frac{2}{13}x^{13/2} + 6x^{1/2}$.

2. (i) $\frac{1}{2}x + \frac{1}{4}\sin 2x$. (ii) $\tan x - x$. (iii) $\frac{1}{2}\sin 2x$. (iv) $\tan x - x$.

3. $\tan x - \sec x$. 4. (i) $\frac{1}{2}x + \frac{\sin 2ax}{4a}$. (ii) $-\cot x - x$.

5. (i) $-2e^{-x} + 3e^x - \frac{4}{3}e^{-3x}$. (ii) $\frac{1}{4}e^{4x}$. (iii) $\frac{1}{3}x^3$.

6. $\frac{x^{a+1}}{a+1} + \frac{x^a}{\log a}$. 7. (i) $\frac{2}{5}x^{5/2} - \frac{2}{9}x^{3/2} + 22x^{1/2}$.

(ii) $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7 + \frac{1}{8}x^8$.

8. (i) $a^2 x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3$.

(ii) $\frac{1}{12}x(3x^3 + 32x^2 + 126x + 216)$.

9. (i) $-\frac{1}{\log a} \left[\frac{a^{-3x}}{3} + \frac{a^{-2x}}{2} + \frac{a^{-x}}{1} \right]$. (ii) $\frac{4}{\log 2} \left[2^{2x} - \frac{1}{3} \cdot 2^{-3x} \right]$.

10. (i) $-3x^{-4/3} - \frac{12}{5}x^{5/3} + \frac{12}{11}x^{-11/3}$. (ii) $a \sec x - b \operatorname{cosec} x$.

(iii) $x + \frac{1}{4}\cos 2x$.

11. (i) $\sin x - \operatorname{cosec} x$. (ii) $\frac{180}{\pi} \sin x$.
12. $-\cot x + \sec x - \cos x$. 13. (i) $\frac{1}{2}x^2 - 2x$. (ii) $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 3x$.
14. (i) x . (ii) $x + 2\sin x$. (iii) $\tan x - x$. (iv) $\sinh x - \cosh x$.
15. (i) $2(\sin \frac{1}{2}x - \cos \frac{1}{2}x)$. or $(2\sqrt{1-\sin x})$. (ii) $2\sqrt{1+\sin x}$.
16. $\sin 2x$. 17. (i) $2\sqrt{2} \sin \frac{1}{2}x$. (ii) $-2\sqrt{2} \cos \frac{1}{2}x$.
- (iii) $-\frac{1}{3}\cos 3x$. 18. (i) $\tan x - \sec x$. (ii) $-\cot x + \operatorname{cosec} x$.
19. (i) $2(\sin x + x \cos a)$. (ii) $-(\sin x + \frac{1}{2} \sin 2x)$
20. (i) $\tan x - \cot x$. (ii) $\tan x - \cot x - 3x$. (iii) $-\frac{1}{2} \sin 2x$. (iv) $\frac{1}{\sqrt{2}}x$
21. $-\operatorname{cosec} x + 3 \cot x + \frac{9}{2}x + \frac{3}{4} \sin 2x$.
22. (i) $\frac{1}{12} \sin 3x + \frac{3}{4} \sin x$. (ii) $\frac{1}{32} \sin 4x - \frac{1}{4} \sin 2x + \frac{3}{8}x$.
23. (i) $\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$. (ii) $\frac{1}{10} \sin 5x + \frac{1}{2} \sin x$.
24. (i) $\frac{1}{8}x - \frac{1}{32} \sin 4x$. (ii) $\frac{1}{4}(\sin 2x - x - \frac{1}{4} \sin 4x)$.
25. $-\frac{1}{8} \cos 2x - \frac{1}{16} \cos 4x + \frac{1}{24} \cos 6x$.