ANSWERS

1. (i) $e^{\tan^{-1}x}$. (ii) $a^{-1}e^{a\sin^{-1}x}$. (iii) $\sin(\log x)$

(iv) $-\frac{1}{3}\cot^3 x$. (v) $\frac{1}{3}\log\cos 3x$. (vi) $\log\sin x$.

2. (i) $\frac{1}{3}(x^2+1)^{3/2}$. (ii) $\frac{2}{9}(a^3+x^3)^{3/2}$.

3. (i) $2\sin\sqrt{x}$. (ii) $\tan\frac{1}{2}x$. **4.** (i) $2\sqrt{\tan x}$. (ii) $\frac{2}{3}\tan^{3/2}x$.

5. (i) $\frac{3}{2}(x+\sin x)^{2/3}$. (ii) $\log(x+\sin x)$.

6. (i) $\log\sec(\log x)$. (ii) $\log(\log x)$. **7.** (i) $2\sqrt{1+\sin x}$.

(ii) $\frac{-1}{b(a+b\sin x)}$. **8.** (i) $-\frac{\sqrt{1-x^2}}{x}$. (ii) $-\frac{\sqrt{1+x^2}}{x}$.

(iii) $x/\sqrt{1-x^2}$. (iv) $x/\sqrt{x^2+1}$.

9. (i) $2\log(e^{x/2}+e^{-x/2})$. (ii) $-\log(1+e^{-x})$. **10.** (i) $-\log(\log\cos x)$.

(ii) $\log(\log\sin x)$. (iii) $\log(\log\tan x)$. (iv) $\log\{\log(\sec x+\tan x)\}$.

11. (i) $\frac{1}{a-b}\log(a\sin^2 x+b\cos^2 x)$.

(ii) $\frac{1}{2(a-b)}\log(a\cos^2 x+b\sin^2 x)$.

(iii) $\frac{1}{(a^2-b^2)}\left\{\frac{1}{a^2\cos^2 x+b^2\sin^2 x}\right\}$. (iv) $-\frac{1}{2b^2}\left\{\frac{1}{a^2+b^2\tan^2 x}\right\}$.

12. (i) $-\frac{1}{2}\cos x^2$. (ii) $\log\tan x$. **13.** (i) $\sqrt{3x^2-2x+7}$.

(ii) $\sqrt{x^2-a^2}$. **14.** (i) $2\sqrt{3+\tan^{-1}x}$. (ii) $2\tan^{1/2}x+\frac{2}{5}\tan^{5/2}x$.

15. (i) $e^x-\log(e^x+1)$. (ii) $\tan^{-1}(e^x)$.

(iii) $x-\log(e^x-1)-(e^x-1)^{-1}$. (iv) $2\tan^{-1}\{\sqrt{(e^x-1)}\}$.

16. $\tan(xe^x)$. **17.** (i) $\frac{1}{3}\{(x+1)^{3/2}+(x-1)^{3/2}\}$.

(ii) $\frac{1}{24}\{(2x+5)^{3/2}-(2x-3)^{3/2}\}$. (iii) $\sec^{-1}(2x-7)$.

18. (i) $2\log(1+\sqrt{x})$. (ii) $-(4x+1)/\{8(2x+1)^2\}$.

19. (i) $2\sqrt{x}+2\log(\sqrt{x}-1)$.

(ii) $\frac{2}{3}x^{3/2}-x+2\sqrt{x}-2\log(\sqrt{x}+1)$.

20.(i) $\frac{3}{10}(2x+1)^{5/2}+\frac{1}{6}(2x+1)^{3/2}$. (ii) $\frac{3}{7}(x+a)^{7/3}-\frac{3}{4}a(x+a)^{4/3}$.

21. (i) $-x-2\log(1-x)$.

(ii) $\frac{1}{6}x^6+\frac{1}{5}x^5+\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x+\log(x-1)$.

22.(i) $\frac{2}{3}x+\frac{1}{9}\log(3x+4)$. (ii) $b^{-2}[(a+bx)-a\log(a+bx)]$.

23.(i) $\frac{4}{27}(3x+2)^{3/2}-\frac{2}{9}(3x+2)^{1/2}$.

(ii) $\frac{3}{5}b^{-2}(a+bx)^{5/3}-\frac{3}{2}ab^{-2}(a+bx)^{2/3}$.

24. $-a\cos^{-1}(x/a)-\sqrt{a^2-x^2}$. **25.** $\frac{1}{3}x^3+\frac{1}{2}x^3+\frac{3}{2}x+\frac{7}{4}\log(2x+1)$.

26.(i) $\frac{2}{3}\sin^{-1}(x/a)^{\frac{3}{2}}$. (ii) $\frac{1}{3}\sin^{-1}(x/a)^3$.

(iii) $\frac{1}{2}\left\{\sec^{-1}x+\frac{1}{x^2}\sqrt{x^2-1}\right\}$. (iv) $-\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{3/2}$.

27. $a\sin^{-1}\left(\frac{x}{a}\right)^{1/2}-\sqrt{x(a-x)}$. **28.** (i) $\frac{x}{a^2\sqrt{(a^2-x^2)}}$.

(ii) $\sqrt{\frac{1+x}{1-x}}$. **29.** (i) $e^{x+\frac{1}{x}}$. (ii) $-x/(x^2-1)$.

30. $\log(a\sin x+b\cos x+c)$. **31.** (i) $b^{-1}\log(a+b\log x)$

(ii) $\frac{1}{3}(\log\sec x)^3$ **32.** $\frac{1}{2}(x^3+3x+6)^{2/3}$ **33.** (i) $\sin(\sin x)$.

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34. $-\frac{1}{2}x^2$. **35.** (i) $x\cos\alpha-\sin\alpha\log\cos(x+\alpha)$.

(ii) $x\cos 2\alpha-\sin 2\alpha\log\sin(x+\alpha)$.

36. (i) $\dfrac{2b}{a^2}\log\dfrac{x}{a-bx}-\dfrac{(a-2bx)}{a^2x(a-bx)}$. (ii) $\dfrac{1}{4}\left\{\log(1-x^4)+\dfrac{1}{1-x^4}\right\}$.

37. (i) $\dfrac{4}{3}\{x^{3/4}-\log(1+x^{3/4})\}$. (ii) $-\sqrt{(1+x^2)^3}/3x^3$.

38. (i) $\dfrac{1}{2}\sec^{-1}x^2$. (ii) $\sqrt{(x^2+1)}/\sqrt{(x^2-1)}$. **39.** $f(x)\phi(x)$.

40. $-\log(1-x^4)$.

2.3. Standard Integrals.

(A) $\displaystyle\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}.(a\neq 0)$

Proof. Put $x=a\tan\theta$; then $dx=a\sec^2\theta\,d\theta$.

$$\therefore\ I=\int\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}=\frac{1}{a}\int d\theta=\frac{1}{a}\theta=\frac{1}{a}\tan^{-1}\frac{x}{a}.$$

Cor. $\displaystyle\int\frac{dx}{1+x^2}=\tan^{-1}x$.

Note. Putting $x=a\cot\theta$, the above integral takes up the form $-\left(\dfrac{1}{a}\right)\cot^{-1}\left(\dfrac{x}{a}\right)$, which evidently differs from the previous form by a constant. Usually

$$\int-\frac{dx}{a^2+x^2}\ \text{is written in the form}\ \frac{1}{a}\cot^{-1}\frac{x}{a}.$$

(B) $\displaystyle\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\ (|x|\neq|a|)$.

Proof. $\displaystyle\int\frac{dx}{x^2-a^2}=\int\frac{1}{2a}\left\{\frac{1}{x-a}-\frac{1}{x+a}\right\}dx=\frac{1}{2a}\left\{\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right\}$

$$=\frac{1}{2a}\{\log|(x-a)|-\log|(x+a)|\}.$$

since the numerator is the differential coefficient of the denominator in each case, [*see Ex.5, Art.* 2.2.],

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Proof. $\int\frac{dx}{x^2-a^2}=\int\frac{1}{2a}\left\{\frac{1}{x-a}-\frac{1}{x+a}\right\}dx=\frac{1}{2a}\left\{\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right\}$

$$=\frac{1}{2a}\{\log|(x-a)|-\log|(x+a)|\}.$$

since the numerator is the differential coefficient of the denominator in each case, [*see Ex.5, Art.* 2.2.],

$$=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|.$$

Note. The above is an example of integration by breaking up the integra into frabtions. [*see Chapter* V.]

(C) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| \quad (|x| \neq |a|)$.

The proof is as before, noticing that

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x} \right).$$

(D) $\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log \left| (x + \sqrt{x^2 \pm a^2}) \right|$

Proof. Put $\sqrt{(x^2 \pm a^2)} = z - x$, or, $z = x + \sqrt{(x^2 \pm a^2)}$.

$$\therefore \quad dz = \left(1 + \frac{2x}{2\sqrt{(x^2 \pm a^2)}} \right) dx = \frac{z}{\sqrt{(x^2 \pm a^2)}}\, dx$$

$$\therefore \quad \int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \int \frac{dz}{z} = \log z = \log \left| x + \sqrt{(x^2 \pm a^2)} \right|.$$

Note. Students acquainted with hyperbolic functions may work out the integrals (D) by substitution $x = a \sin h\, z$, or $x = a \cosh z$ according as the denominator is $\sqrt{(x^2 + a^2)}$ or $\sqrt{(x^2 - a^2)}$.

Thus, putting $x = a \sinh z$, we have $dx = a \cosh z\, dz$.

(1) Hence, $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \cosh z}{a\sqrt{1 + \sinh^2 z}}\, dz = \int dz = z = \sinh^{-1} \frac{x}{a}$ and this is shown in Trigonometry*

$$= \log \left| \frac{x + \sqrt{x^2 + a^2}}{a} \right| = \log \left| (x + \sqrt{x^2 + a^2} \right| - \log a.$$

This form differs from the result given above by a constant [*Cf Remarks in Art*. 1.3]. In the result, $-\log a$ being a constant may be dropped, since it may be supposed to be included in the constant of integration.

* See *Das & Mukherjee's Higher Trigonometry, Art 12.9.*

In that case the forms given in (D) and here are the same. Similar remark applies to the results of (2) and (3) below.

Similarly, by putting $x = a\cosh z$, we have

(2) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} = \log\left|\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)\right|$.

In many text-books, for the sake of brevity, inverse hyperbolic forms are used in preference to the logarithmic forms.

The first of the integrals (*D*) can also be evaluated by putting $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$. Thus,

(3) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta} = \int \sec\theta\, d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\, d\theta$

$= \log(\sec\theta + \tan\theta)$ [*by Ex.* 5, *Art.* 2.2]

$= \log\left(\sqrt{(1 + \tan^2\theta)} + \tan\theta\right)$

$= \log\left(\sqrt{1 + \frac{x^2}{a^2}} + \frac{x}{a}\right) = \log\left|\frac{x + \sqrt{x^2 + a^2}}{a}\right|$.

Similarly, put $x = a\sec\theta$, in the other integral.

Similarly the integrals of (*B*) and (*C*) can be obtained by hyperbolic substitution.

(B) Thus, putting $x = a\coth\theta$, $dx = -a\,\mathrm{cosec}h^2\theta\, d\theta$ and

$$(x^2 - a^2) = a^2(\coth^2\theta - 1) = a^2\,\mathrm{cosec}h^2\theta$$

$$\therefore \quad I = \int \frac{-a\,\mathrm{cosech}^2\theta}{a^2\,\mathrm{cosech}^2\theta}\, d\theta = -\frac{1}{a}\int d\theta = -\frac{1}{a}\theta = -\frac{1}{a}\coth^{-1}\frac{x}{a}.$$

(C) Similarly, putting $x = a\tanh\theta$,

$$I = \int \frac{a\,\mathrm{sech}^2\theta\, d\theta}{a^2\,\mathrm{sech}^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tanh^{-1}\frac{x}{a}.$$

(E) $\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a}$. $(|x| < |a|)$

Put $x = a\sin\theta$; then $dx = a\cos\theta\, d\theta$.