$$= n\left\{\frac{u^{n-1}}{n-1} - {}^{n-1}C_1\frac{u^{n-2}}{n-2} + \cdots \cdots + (-1)^{n-1}\log|u|\right\} + C$$

$$= n\left\{\frac{\left(1+\sqrt[n]{x+1}\right)^{n-1}}{n-1} - {}^{n-1}C_1\frac{\left(1+\sqrt[n]{x+1}\right)^{n-2}}{n-2} + \cdots \cdots \right.$$

$$\left. + (-1)^{n-1}\log\left|1+\sqrt[n]{x+1}\right|\right\} + C$$

Ex. 16 Integrate $\int x^3 \tan^4 x^4 \sec^2 x^4 dx$.

Solution : Let $I = \int x^3 \tan^4 x^4 \sec^2 x^4 dx$

We substitute $\tan^4 x = u$

then $\sec^2 x^4 .4x^3\, dx = du \Rightarrow x^3 .\sec^2 x^4 dx = \frac{1}{4} du$

So, $I = \frac{1}{4}\int u^4 du = \frac{1}{4}.\frac{1}{5}u^5 + C = \frac{1}{20}\tan^5(x^4) + C$.

Ex. 17 Find $\int \frac{x^2 - 1}{(x^4 + 3x^2 + 1)\tan^{-1}\left(1 + \frac{1}{x}\right)} dx$.

Solution : Here $I = \int \frac{1 - \frac{1}{x^2}}{\left(x^2 + 3 + \frac{1}{x^2}\right)\tan^{-1}\left(x + \frac{1}{x}\right)} dx$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left\{\left(x + \frac{1}{x}\right)^2 + 1\right\}\tan^{-1}\left(x + \frac{1}{x}\right)} dx$$

Let us substitute $x + \frac{1}{x} = u \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = du$.

Then $I = \int \frac{du}{(u^2+1)\tan^{-1}(u)}$

Now, we put $\tan^{-1} u = z \Rightarrow \frac{du}{1+u^2} = dz$

Hence $I = \int \frac{dz}{z} = \log|z| + C$

$= \log|\tan^{-1} u| + C = \log\left|\tan\left(1+\frac{1}{x}\right)\right| + C$.

Ex. 18 Evaluate : $\int \frac{10x^9 + 10^x . \log_e 10}{\sqrt{10^x + x^{10} + 7}} dx$.

Solution : Let us put

$10^x + x^{10} + 7 = u \Rightarrow (10^x \log_e 10 + 10x^9)\, dx = du$

Then, $I = \int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}} du = 2u^{\frac{1}{2}} + C = 2\sqrt{10^x + x^{10} + 7} + C$.

Ex. 19 Find : $\int \sqrt{\frac{x-a}{b-x}}\, dx$.

Solution : We put $x = a\cos^2\theta + b\sin^2\theta$

Then $dx = 2(b-a)\sin\theta\cos\theta\, d\theta$.

Also, $x - a = (b-a)\sin^2\theta$, $b - x = (b-a)\cos^2\theta$.

Hence, $I = \int \frac{\sin\theta}{\cos\theta} . 2(b-a)\sin\theta\cos\theta\, d\theta$

$= (b-a)\int 2\sin^2\theta\, d\theta$

$= (b-a)\int (1-\cos 2\theta)\, d\theta$

$= (b-a)\left\{\theta - \frac{1}{2}\sin 2\theta\right\} + C$

$= (b-a)[\theta - \sin\theta\cos\theta] + C$

$$= (b-a)\left[\tan^{-1}\sqrt{\frac{x-a}{b-x}} - \frac{\sqrt{(x-a)(b-x)}}{(b-a)}\right] + C.$$

Ex. 20 Find $\int \frac{(x^4-1)}{x^2\sqrt{(x^4+x^2+1)}}dx$

Solution : $I = \int \frac{(x^4-1)}{x^2 \cdot x\sqrt{x^2+1+\frac{1}{x^2}}}dx = \int \frac{(x-\frac{1}{x^3})}{\sqrt{x^2+1+\frac{1}{x^2}}}dx$

$= \int \frac{dt}{2\sqrt{t}}$, where $t = x^2+1+\frac{1}{x^2}$

$= \frac{1}{2}\int t^{-\frac{1}{2}}dt = \frac{1}{2}.2t^{\frac{1}{2}} + c = \sqrt{t} + C$

$= \sqrt{x^2+1+\frac{1}{x^2}} + C = \frac{1}{x}\sqrt{x^4+x^2+1} + C.$

21. Find $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$.

Solution : We put $x^{\frac{3}{2}} = Z$, then $x^3 = Z^2$ and

$\frac{3}{2}x^{\frac{1}{2}}dx = dZ \Rightarrow \sqrt{x}\,dx = \frac{2}{3}dZ$.

So, $I = \frac{2}{3}\int \frac{dZ}{\sqrt{a^3-Z^2}} = \frac{2}{3}\int \frac{dZ}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - Z^2}} = \frac{2}{3}.\sin^{-1}\left(\frac{Z}{a^{\frac{3}{2}}}\right) + C$

$= \frac{2}{3}.\sin^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}} + C.$

Solution : Here $I=\int (x-1)^{-\frac{3}{4}}(x+2)^{-\frac{5}{4}}\,dx$.

$$=\int (x-1)^{-1+\frac{1}{4}}.(x+2)^{-1-\frac{1}{4}}\,dx$$

$$=\int\left(\frac{x-1}{x+2}\right)^{-1+\frac{1}{4}}\cdot\frac{1}{(x+2)^2}\,dx$$

If we put $t=\frac{x-1}{x+2}$, then $dt=\frac{1(x+2)-1(x-1)}{(x+2)^2}.dx$

$$\Rightarrow \frac{1}{3}dt=\frac{dx}{(x+2)^2}$$

Therefore, $I=\frac{1}{3}\int t^{-\frac{3}{4}}dt=\frac{1}{3}.4t^{\frac{1}{4}}+C=\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}}+C$.

23. Find $\int \frac{x}{1+x\tan x}dx$.

Solution : $I=\int\frac{x\cos x\,dx}{\cos x+x\sin x}dx$

$=\int\frac{du}{u}$, where $u=\cos x+x\sin x$

$\Rightarrow du=(-\sin x+\sin x+x\cos x)\,dx$

$=x\cos x\,dx$

$=\log|u|+C=\log|\cos x+x\sin x|+C$.

24. Find $\int\frac{dx}{\sin(x-\alpha)\cos(x-\beta)}$.

Solution : $I=\frac{1}{\cos(\beta-\alpha)}.\int\frac{\cos\{(x-\alpha)-(x-\beta)\}}{\sin(x-\alpha)\cos(x-\beta)}dx$

$$=\frac{1}{\cos(\beta-\alpha)}.\int\frac{\cos(x-\alpha)\cos(x-\beta)+\sin(x-\alpha)\sin(x-\beta)}{\sin(x-\alpha)\cos(x-\beta)}dx$$

$$=\frac{1}{\cos(\beta-\alpha)}\int\left\{\frac{\cos(x-\alpha)}{\sin(x-\alpha)}+\frac{\sin(x-\beta)}{\cos(x-\beta)}\right\}dx$$

$$=\frac{1}{\cos(\beta-\alpha)}\{\log|\sin(x-\alpha)|-\log|\cos(x-\beta)|\}+C$$

$$=\sec(\alpha-\beta)\log|\tan(x-\alpha)|+C.$$

25. Find $\int\{1+\tan x\tan(x+\alpha)\}dx$.

Solution : Here $I=\int\left\{1+\frac{\sin x\sin(x+\alpha)}{\cos x\cos(x+\alpha)}\right\}dx$

$$=\int\left\{\frac{\cos x\cos(x+\alpha)+\sin x\sin(x+\alpha)}{\cos x\cos(x+\alpha)}\right\}dx$$

$$=\cos\alpha\int\frac{1}{\cos x\cos(x+\alpha)}dx$$

$$=\frac{\cos\alpha}{\sin\alpha}\int\frac{\sin\{(x+\alpha)-x\}}{\cos x\cos(x+\alpha)}dx$$

$$=\cot\alpha\int\frac{\sin(x+\alpha)\cos x-\cos(x+\alpha)\sin x\}}{\cos x\cos(x+\alpha)}dx$$

$$=\cot\alpha\int\left\{\frac{\sin(x+\alpha)}{\cos(x+\alpha)}-\frac{\sin x}{\cos x}\right\}dx$$

$$=\cot\alpha\{-\log|\cos(x+\alpha)|+\log(\cos x)\}+C$$

$$=\cot\alpha\log\left|\frac{\cos x}{\cos(x+\alpha)}\right|+C.$$

EXAMPLES II (B)

Integrate :

1. $\int\frac{3x^2}{1+x^6}dx$.

2. (i) $\int\frac{xdx}{x^4+1}$. (ii) $\int\frac{xdx}{x^4-1}$.

3. (i) $\int\frac{dx}{e^x+e^{-x}}$. [Put $e^x=z$.] (ii) $\int\frac{x^3dx}{\sqrt{a^8-x^8}}$. [Put $x^4=z$.]

4. (i) $\int \frac{x^2+\sin^2 x}{1+x^2}\sec^2 x dx$. (ii) $\int \frac{\sin x\, dx}{3+\sin^2 x}$.

5. (i) $\int \frac{x dx}{\sqrt{(a^4+x^4)}}$. (ii) $\int \frac{x^2-1}{x\sqrt{(1+x^4)}} dx$.

[(ii) Put $x+x^{-1}=z$.]

6. $\int \frac{x dx}{\sqrt{\{(x^2-a^2)(b^2-x^2)\}}}$ $(b^2>a^2)$. [Put $x^2-a^2=z^2$.]

7. (i) $\int \frac{dx}{1+x+x^2}$. (ii) $\int \frac{dx}{4x^2+4x+5}$.

8. (i) $\int \frac{dx}{1+x-x^2}$. (ii) $\int \frac{dx}{6x^2+7x+2}$.

9. $\int \frac{x dx}{x^4+2x^2+2}$.

10. $\int \frac{\cos x dx}{\sin^2 x+4\sin x+3}$.

11. $\int \frac{e^x dx}{e^{2x}+2e^x+5}$.

12. $\int \frac{dx}{\sqrt{(1-x^2)\{1+(\sin^{-1} x)^2\}}}$.

13. $\int \frac{x^2 dx}{x^6-6x^3+5}$.

14. $\int \frac{dx}{x\{10+7\log x+(\log x)^2\}}$.

15. (i) $\int \frac{x dx}{x^2+2x+1}$. (ii) $\int \frac{x+1}{3+2x-x^2} dx$.

16. (i) $\int \frac{x+1}{x^2+4x+5} dx$. (ii) $\int \frac{2x+3}{4x^2+1} dx$.

17. (i) $\int \frac{(4x+3) dx}{3x^2+3x+1}$. (ii) $\int \frac{x dx}{2-6x-x^2}$.

18. $\int \frac{x^2}{x^2-4} dx$.

19. (i) $\int \frac{x^2+2x}{x^2+2x+2} dx$. (ii) $\int \frac{x^2-x+1}{x^2+x+1} dx$.

20. $\int \frac{x^3+x^2+2x+1}{x^2-x+1}\,dx$.

21. $\int \frac{dx}{\sqrt{(x^2+x-2)}}$.

22. (i) $\int \frac{dx}{\sqrt{(1-x-x^2)}}$.

(ii) $\int \frac{dx}{\sqrt{(3+3x+x^2)}}$.

23. $\int \frac{dx}{\sqrt{(2x^2+3x+4)}}$.

24. $\int \frac{dx}{\sqrt{(x^2-7x+12)}}$. [Put $x-4=z^2$]

25. $\int \frac{dx}{\sqrt{(6+11x-10x^2)}}$.

26. $\int \frac{\cos x\,dx}{\sqrt{(5\sin x^2 x-12\sin x+4)}}$.

27. $\int \frac{dx}{\sqrt{\{(x-\alpha)(x-\beta)\}}}$.

28. (i) $\int \frac{dx}{\sqrt{(2ax-x^2)}}$.

(ii) $\int \frac{dx}{\sqrt{(2ax+x^2)}}$.

29. (i) $\int \frac{x+b}{\sqrt{(x^2+a^2)}}\,dx$.

(ii) $\int \frac{2x+3}{\sqrt{(x^2+x+1)}}\,dx$.

30. $\int \frac{x-2}{\sqrt{(2x^2-8x+5)}}\,dx$.

31. (i) $\int \frac{(x+1)}{\sqrt{(4+8x-5x^2)}}\,dx$.

(ii) $\int \frac{(2x-1)\,dx}{\sqrt{(4x^2+4x+2)}}\,dx$.

32. (i) $\int \frac{dx}{(2x-1)\sqrt{(1+x)}}$.

(ii) $\int \frac{dx}{(2x+1)\sqrt{(4x+3)}}$.

33. $\int \frac{dx}{\sqrt{\frac{2}{3}x^3-x^2+\frac{1}{3}}}$.

34. (i) $\int \sqrt{\left(\frac{x-3}{x-4}\right)}\,dx$.

(ii) $\int \sqrt{\left(\frac{2x+1}{3x+2}\right)}\,dx$.